Determine if each of the following converges or diverges.
If it converges, write "CONVERGES". If it diverges, write "DIVERGES".
Justify each answer using proper mathematical reasoning, algebra and/or calculus.

ALL HIGHLIGHTED ITEMS WORTH ① POINT UNLESS OTHERWISE NOTED

[a] $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{e^{\frac{1}{n}}}$

SCORE: _____ / 4 PTS

$$\lim_{n\to\infty} \frac{\tan^{-1} n}{e^{\frac{1}{n}}} = \frac{\frac{\pi}{2}}{e^0} = \frac{\pi}{2} \neq 0$$

$\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{e^{\frac{1}{n}}}$ DIVERGES BY DIVERGENCE TEST

[b] $\displaystyle\sum_{n=1}^{\infty} \frac{500n-4}{n+n^2 3^n}$

SCORE: _____ / $5\frac{1}{2}$ PTS

$\frac{1}{2}$ $$0 < \frac{500n-4}{n+n^2 3^n} < \frac{500n}{n^2 3^n} = \frac{500}{n3^n} < \frac{500}{3^n}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{500}{3^n}$ CONVERGES ($|r| = \frac{1}{3} < 1$) BY GEOMETRIC SERIES TEST

SO $\displaystyle\sum_{n=1}^{\infty} \frac{500n-4}{n+n^2 3^n}$ CONVERGES BY COMPARISON TEST

[c] $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^{10}+84n^2+506}}{(8n^2+4n+1)^2}$

SCORE: ____ / 8 PTS

LET $b_n = \dfrac{\sqrt[3]{n^{10}}}{(8n^2)^2} = \dfrac{1}{64} \cdot \dfrac{1}{n^{\frac{2}{3}}}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left| \frac{\sqrt[3]{n^{10}+84n^2+506}}{(8n^2+4n+1)^2} \cdot \frac{(8n^2)^2}{\sqrt[3]{n^{10}}} \right| = \lim_{n\to\infty} \left| \sqrt[3]{\frac{n^{10}+84n^2+506}{n^{10}}} \cdot \left(\frac{8n^2}{8n^2+4n+1}\right)^2 \right|$$

$$= \sqrt[3]{1} \cdot 1^2 = 1 \neq 0$$

$\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{64} \cdot \frac{1}{n^{2/3}}$ DIVERGES $(p = \frac{2}{3} < 1)$ BY p-SERIES TEST

SO $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^{10}+84n^2+506}}{(8n^2+4n+1)^2}$ DIVERGES BY LIMIT COMPARISON TEST

[d] $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$

SCORE: ____ / 7 PTS

$\dfrac{1}{x(\ln x)^4}$ IS CONTINUOUS ON $[2,\infty)$

$\frac{1}{2}$ >0 SINCE $x, \ln x > 0$ ON $[2,\infty)$

$\frac{1}{2}$ AND DECREASING ON $[2,\infty)$

SINCE $\dfrac{d}{dx}\left(\dfrac{1}{x(\ln x)^4}\right) = \dfrac{-((\ln x)^4 + (\ln x)^3)}{x^2(\ln x)^8} < 0$ ON $[2,\infty)$

$$\int_2^{\infty} \frac{1}{x(\ln x)^4}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^4}\,du = \lim_{N\to\infty} -\frac{1}{3u^3}\Big|_{\ln 2}^{\infty} \text{ OR } \lim_{x\to\infty} -\frac{1}{3(\ln x)^3}\Big|_2^{\infty}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{N\to\infty} -\frac{1}{3}\left(\frac{1}{N^3} - \frac{1}{(\ln 2)^3}\right) \text{ OR } \lim_{x\to\infty} -\frac{1}{3}\left(\frac{1}{(\ln x)^3} - \frac{1}{(\ln 2)^3}\right)$$

$$= \frac{1}{3(\ln 2)^3} \text{ CONVERGES}$$

SO $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ CONVERGES BY INTEGRAL TEST

[e] $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$

SCORE: ____ / $5\frac{1}{2}$ PTS

$\frac{1}{2}$ $$0 < \frac{4^n}{n!} = \frac{4\cdot4\cdot4\cdot4\cdot4\cdot4\cdots4}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdots n} < \frac{4\cdot4\cdot4\cdot4\cdot4\cdot4\cdots4}{1\cdot2\cdot3\cdot4\cdot5\cdot5\cdots5} = \frac{4^4}{4!}\left(\frac{4}{5}\right)^{n-4}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{4^4}{4!}\left(\frac{4}{5}\right)^{n-4}$ CONVERGES $(|r| = \frac{4}{5} < 1)$ BY GEOMETRIC SERIES TEST

SO $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{4!}$ CONVERGES BY COMPARISON TEST